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Analysis of Heterogeneous Reactors Containing Moderating Fuel Elements

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CONTENTS

FIGURES

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ABSTRACT

The Feinberg-Galanin method for heterogeneous reactors is formulated by using a two-group model rather than an age kernel. This treatment is then extended to take into account secondary effects, such as fast fission and thermalization of neutrons inside a rod which may contain moderator. The use of a single coefficient in a Feinberg-Galanin approach allows one to relate the source and sink strength of the fuel element to the thermal flux only. By defining a set of four coefficients, it is possible to connect the strengths of thermal- and fast-neutron sources and sinks to both thermal and fast fluxes. A method is presented for calculation of these four coefficients, α_1, β_1, α_2, and β_2.

I. INTRODUCTION

The theory of heterogeneous reactors was first developed with the homogeneous model by calculating the criticality of equivalent homogeneous reactors. The actual reactor is divided into cells whose shape is determined by the fuel-element distribution. Using the Wigner-Seitz unit-cell model, each cell is replaced by an equivalent (usually cylindrical) cell whose nuclear parameters can be calculated. The heterogeneous cells are then replaced in the reactor model by homogeneous regions having these calculated nuclear characteristics.

Subsequently, Feinberg (Ref. 1) and Galanin (Ref. 2) developed a heterogeneous method applicable to infinite-moderator media. This method was extended to finite media of rectangular shape by Meetz (Ref. 3) and to those of cylindrical shape by Jonsson (Ref. 4).

Feinberg and Galanin consider each fuel rod as a singularity. These singularities are considered as external to the moderator and are treated as localized sources and sinks. The properties of the rods are contained in a single constant γ, which relates the thermal net current to the thermal flux at the surface of the fuel element; this allows one to relate the strength of the thermal-neutron sink at the rod to the thermal-neutron flux at its surface. The singularity is then considered as a sink of thermal neutrons and a source of fast neutrons. The strength of the source is related to the strength of the sink through the coefficient η, the average number of fission neutrons produced per neutron absorbed in the rod. This coefficient may be different for each rod.

For reactors with a small number of solid rods, the Feinberg-Galanin method is an improvement over the homogeneous method. However, it is not completely satisfactory for reactors which may have fuel elements containing a significant amount of moderator. In such reactors, fast neutrons can be produced in a fuel element by fission, but slow neutrons can also be produced by a slowing-down inside a fuel element. Neither of the two methods noted above treats this kind of reactor properly; the homogeneous method does not localize the singularity, and the Feinberg-Galanin method neglects the slowing-down inside the singularity.

In order to introduce these effects in a heterogeneous-reactor calculation, the Feinberg-Galanin method is reconstructed in a two-group model (Section II), after which the additional source and sink effects in the fast and thermal groups are introduced (Section III). Thus, each fuel element is represented by:

(1) a source of fast neutrons (fission)

(2) a source of thermal neutrons (thermalization)

(3) a sink of fast neutrons (radiative capture and fast fission)

(4) a sink of thermal neutrons (thermal absorption)

The source and sink terms are related to each other by two constants: η_1 and η_2, the average number of neutrons produced per fast and thermal absorption, respectively, in the fuel element; and by p, the probability that a neutron slowing down inside the fuel element reaches thermal energy.

The change introduced here with respect to the Feinberg-Galanin method is that the sink terms are related to both the thermal and the fast fluxes at the surface of the rod. These relationships necessitate the establishment of four coefficients, which must be determined. Section IV outlines a method of obtaining these coefficients for a simple fuel element.

Although the treatment here is for the case of fuel rods only, the extension to control rods is straightforward. In that case, the coefficients η are equal to zero.

II. FEINBERG-GALANIN METHOD WITH TWO-GROUP MODEL

Consider an infinite moderating medium containing a finite number of fuel elements. These fuel elements are assumed to be cylinders of finite length and parallel to each other, forming a core embedded in the moderator. It is assumed either that the distance between these elements is large compared with their transverse dimensions, or that the lattice is sufficiently symmetrical that the flux near the elements possesses enough symmetry for consideration of the elements as line sources. In order that diffusion theory may be used in the moderator, the further assumption is made that the distance between two fuel elements is large compared with the diffusion length.

Applying diffusion theory, one can write equations for the overall fluxes:

$$-D_1 \nabla^2 \phi_1(\boldsymbol{\rho}) + (\Sigma_R + \Sigma_a^{(1)})\, \phi_1(\boldsymbol{\rho}) = \eta \sum_{k=1}^{N} S_k(\boldsymbol{\rho})\, \delta(\mathbf{r} - \mathbf{r}_k) \qquad (1)$$

$$-D_2 \nabla^2 \phi_2(\boldsymbol{\rho}) + \Sigma_a^{(2)}\, \phi_2(\boldsymbol{\rho}) = p\, \Sigma_R\, \phi_1(\boldsymbol{\rho}) - \sum_{k=1}^{N} S_k(\boldsymbol{\rho})\, \delta(\mathbf{r} - \mathbf{r}_k) \qquad (2)$$

where $\phi_1(\boldsymbol{\rho})$ and $\phi_2(\boldsymbol{\rho})$ refer to the fast and thermal fluxes, respectively, in the moderator; Σ_R is the removal cross section of the moderator; and N is the total number of elements.

On the right-hand sides of Eqs. 1 and 2 are the densities of the different neutron sources of the system. It may be seen that each fuel element is considered as a line sink capturing S_k thermal neutrons per unit time and unit length. The quantity S_k depends on the position along the z axis of the fuel element, which is the reason for introducing $S_k(\boldsymbol{\rho})$ in Eqs. 1 and 2. Note that the Direc delta functions used in the Equations are two-dimensional delta functions, so that the product $S_k(\boldsymbol{\rho}) \times \delta(\mathbf{r} - \mathbf{r}_k)$ is the sink density of the fuel element k at point $\boldsymbol{\rho}$.

The geometry of the fuel elements suggests the use of cylindrical coordinates; thus, the variables in the differential equations are separated. Let

$$\phi(\boldsymbol{\rho}) = \phi(\mathbf{r})\,\phi(z) \tag{3}$$

$$\phi(z) \propto \cos B_z z \tag{4}$$

$$B_z = \frac{\pi}{2h + 2\Delta_z} \tag{5}$$

where Δ_z is the reflector saving on one side along the z axis, along which the reflector need not be infinite. The term $S_k(\boldsymbol{\rho})$, which is directly proportional to the flux, can be written as

$$S_k(\boldsymbol{\rho}) = S_k \cos B_z z$$

where S_k is now a constant to be determined for each rod. Remaining are the two-dimensional equations:

$$-\nabla^2 \phi_1(\mathbf{r}) + \kappa_1^2\,\phi_1(\mathbf{r}) = \frac{\eta}{D_1}\sum_{k=1}^{N} S_k\,\delta(\mathbf{r} - \mathbf{r}_k) \tag{6}$$

$$-\nabla^2 \phi_2(\mathbf{r}) + \kappa_2^2\,\phi_2(\mathbf{r}) = \frac{p\Sigma_R}{D_2}\,\phi_1(\mathbf{r}) - \frac{1}{D_2}\sum_{k=1}^{N} S_k\,\delta(\mathbf{r} - \mathbf{r}_k) \tag{7}$$

where

$$\kappa_1^2 = \frac{\Sigma_R + \Sigma_a^{(1)}}{D_1} + B_z^2$$
$$\kappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2} + B_z^2 \tag{8}$$

In the fast-flux equation, the solution sought is of the form:

$$\phi_1(\mathbf{r}) = \sum_{k=1}^{N} A_k\,K_0(\kappa_1\,|\mathbf{r} - \mathbf{r}_k|) \tag{9}$$

which is the solution for a superposition of k line sources located at different positions r_k and satisfies the boundary condition for an infinite-moderator medium.

One can easily find the coefficients A_k by making the following assumption: The flux at a rod emplacement is the sum of a symmetrical rapidly varying function due to the rod itself and an unsymmetrical slowly varying function due to all the other rods. In computing the derivative about a rod, the derivative of the slowly varying function is neglected when compared with the derivative of the other function. The rapidly varying function at the rod k is stated as

$$\phi_k = A_k\,K_0(\kappa_1\,|\mathbf{r} - \mathbf{r}_k|)$$

In order to evaluate the coefficients A_k, the finite radius b of the rod must now be considered. The number of fast neutrons leaving the surface of the rod per unit length and time is given by

$$-D_1\frac{\partial\phi_1}{\partial r}2\pi b = 2\pi b\,D_1\,\kappa_1\,A_k\,K_1(\kappa_1 b) = \eta S_k \tag{10}$$

where S_k is the number of thermal neutrons absorbed per unit length and time at the center of rod k; then

$$A_k = \frac{\eta S_k}{2\pi b\,D_1\,\kappa_1\,K_1(\kappa_1 b)} \tag{11}$$

The thermal constant γ, defined in Appendix A, is expressed as

$$\gamma = \frac{-2\pi b\,J_2(\mathbf{r}_k)}{\phi_2(\mathbf{r}_k)} = \frac{S_k}{\phi_2(\mathbf{r}_k)} \tag{12}$$

where $\phi_2(\mathbf{r}_k)$ and $J_2(\mathbf{r}_k)$ are the thermal flux and net current (taken as positive when directed outward) at the surface of fuel element k. The fast flux is thus expressed by

$$\phi_1(\mathbf{r}) = \frac{\eta\gamma}{2\pi b\,D_1\,\kappa_1\,K_1(\kappa_1 b)}\sum_{k=1}^{N}\phi_2(\mathbf{r}_k)\,K_0(\kappa_1\,|\mathbf{r}-\mathbf{r}_k|) \tag{13}$$

The thermal equation (Eq. 7) is, then,

$$-\nabla^2\phi_2(\mathbf{r}) + \kappa_2^2\,\phi_2(\mathbf{r}) = m\sum_{k=1}^{N}\phi_2(\mathbf{r}_k)\,K_0(\kappa_1\,|\mathbf{r}-\mathbf{r}_k|) - \frac{\gamma}{D_2}\sum_{k=1}^{N}\phi_2(\mathbf{r}_k)\,\delta(\mathbf{r}-\mathbf{r}_k) \tag{14}$$

where

$$m = p\frac{\Sigma_R}{D_2}\,\frac{\eta\gamma}{2\pi b\,D_1\,\kappa_1\,K_1(\kappa_1 b)}$$

Applying a Fourier transformation to Eq. 14, inverting and integrating the result, one finds the following expression for the flux:

$$\phi_2(\mathbf{r}) = \sum_{k=1}^{N} \phi_2(\mathbf{r}_k)\, H(|\mathbf{r}-\mathbf{r}_k|) \tag{15}$$

where

$$H(|\mathbf{r}-\mathbf{r}_k|) = \frac{m}{\kappa_2^2 - \kappa_1^2} K_0(\kappa_1 |\mathbf{r}-\mathbf{r}_k|) - \left(\frac{m}{\kappa_2^2 - \kappa_1^2} + \frac{\gamma}{2\pi D_2}\right) K_0(\kappa_2 |\mathbf{r}-\mathbf{r}_k|) \tag{16}$$

III. THE FOUR-COEFFICIENT METHOD

Considered here is the same assembly as the one defined in Section II; now, however, the fuel elements are allowed to have non-negligible slowing-down properties. Each fuel element is replaced by:

(1) a sink of thermal neutrons

(2) a sink of fast neutrons due to absorption and thermalization

(3) a source of fast neutrons due to fission

(4) a source of thermal neutrons due to thermalization inside the rod

To write the balance equations, one relates the source and sink strengths to the coefficients η and p.

A. The Neutron-Balance Equations

One writes the same equations as those used in Section II; but $S_k^{(1)}$ and $S_k^{(2)}$, the fast and thermal neutron-sink terms, now replace the one thermal-sink term:

$$(-D_1\nabla^2 + \Sigma_R + \Sigma_a^{(1)})\,\phi_1(\boldsymbol{\rho}) = \eta_2 \sum_{k=1}^{N} S_k^{(1)}(\boldsymbol{\rho})\,\delta(\mathbf{r}-\mathbf{r}_k) + (\eta_1 - 1)\sum_{k=1}^{N} S_k^{(1)}(\boldsymbol{\rho})\,\delta(\mathbf{r}-\mathbf{r}_k) \tag{17}$$

$$(-D_2\nabla^2 + \Sigma_a^{(2)})\,\phi_2(\boldsymbol{\rho}) = p\,\Sigma_R\,\phi_1(\boldsymbol{\rho}) - \sum_{k=1}^{N} S_k^{(2)}(\boldsymbol{\rho})\,\delta(\mathbf{r}-\mathbf{r}_k) + p\sum_{k=1}^{N} S_k^{(1)}(\boldsymbol{\rho})\,\delta(\mathbf{r}-\mathbf{r}_k) \tag{18}$$

where

$S_k^{(1)}$ is the number of neutrons which disappear from the fast group per unit length and time at the center of rod k.

$S_k^{(2)}$ is the number of thermal neutrons absorbed per unit length and time at the center of rod k.

η_1 is the average number of fast neutrons produced per fast absorption in rod k (may depend on k).

η_2 is the average number of fast neutrons produced per thermal absorption in rod k.

Note the extra terms in the right-hand side of these Equations:

(1) The term $(\eta_1 - 1)\,S_k^{(1)}(\boldsymbol{\rho})\;\delta\,(\mathbf{r}-\mathbf{r}_k)$ accounts for fast absorption and fast fission in rod k.

(2) The term $p\;S_k^{(1)}(\boldsymbol{\rho})\;\delta(\mathbf{r}-\mathbf{r}_k)$ accounts for thermalization in rod k; the overall resonance-escape probability p is used as an approximation.

In these Equations, the z dependency can be removed by letting

$$\phi(\boldsymbol{\rho}) = \phi(\mathbf{r})\cos B_z z$$

Then,

$$(-\nabla^2 + \kappa_1^2)\,\phi_1(\mathbf{r}) = \frac{\eta_2}{D_1}\sum_{k=1}^{N} S_k^{(2)}\delta(\mathbf{r}-\mathbf{r}_k) + \frac{\eta_1 - 1}{D_1}\sum_{k=1}^{N} S_k^{(1)}\delta(\mathbf{r}-\mathbf{r}_k) \tag{19}$$

$$(-\nabla^2 + \kappa_2^2)\phi_2(\mathbf{r}) = \frac{p}{D_2}\Sigma_R\,\phi_1(\mathbf{r}) - \frac{1}{D_2}\sum_{k=1}^{N} S_k^{(2)}\,\delta(\mathbf{r}-\mathbf{r}_k) + \frac{p}{D_2}\sum_{k=1}^{N} S_k^{(1)}\,\delta(\mathbf{r}-\mathbf{r}_k) \tag{20}$$

where

$$\kappa_1^2 = \frac{\Sigma_R + \Sigma_a^{(1)}}{D_1} + B_z^2$$

$$\kappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2} + B_z^2$$

$$B_z = \frac{\pi}{2h + 2\Delta_z} \tag{21}$$

It is now necessary to express $S_k^{(1)}$ and $S_k^{(2)}$ in terms of the fluxes at the surface of each rod $\phi_1(\mathbf{r}_k)$ and $\phi_2(\mathbf{r}_k)$.

Therefore, the coefficients are defined:

α_1 is the probability that a fast neutron entering the fuel element escapes from the fuel element as a fast neutron.

β_1 is the probability that a fast neutron born from fission inside the fuel element escapes from the fuel element as a fast neutron.

α_2 is the probability that a thermal neutron entering the fuel element escapes from the fuel element.

β_2 is the probability that a neutron thermalized inside a fuel element escapes from the fuel element.

Using these coefficients, one can write:

$$S_k^{(1)} = 2\pi b\, j_1^- (1 - \alpha_1) + \eta_2 S_k^{(2)} (1 - \beta_1) + \eta_1 S_k^{(1)} (1 - \beta_1) \tag{22}$$

$$S_k^{(2)} = 2\pi b\, j_2^- (1 - \alpha_2) + p\, S_k^{(1)} (1 - \beta_2) \tag{23}$$

where j^- is the partial current going inward at the rod surface, and b is the radius of the rod.

The fast and thermal net currents can be expressed by

$$2\pi b\, J_1 = 2\pi b\, (j_1^+ - j_1^-) = \eta_2 S_k^{(2)} \beta_1 + S_k^{(1)} (\eta_1 \beta_1 - 1) \tag{24}$$

$$2\pi b\, J_2 = 2\pi b (j_2^+ - j_2^-) = p\, S_k^{(1)} \beta_2 - S_k^{(2)} \tag{25}$$

and, from diffusion theory,

$$j_1^+ + j_1^- = \frac{\phi_1(\mathbf{r}_k)}{2} \tag{26}$$

$$j_2^+ + j_2^- = \frac{\phi_2(\mathbf{r}_k)}{2} \tag{27}$$

where the j's and ϕ's denote the partial currents and fluxes at the outside surface of a rod.

Note that, because of the assumption of symmetry made at the beginning of Section II, the flux at the surface of a rod is independent of the azimuthal angle.

With the six linear relationships stated in Eqs. 22–27, one can find the S_k's in terms of the ϕ's:

$$S_k^{(1)} = a_1 \phi_1(\mathbf{r}_k) + b_1 \phi_2(\mathbf{r}_k) \tag{28}$$

$$S_k^{(2)} = a_2 \phi_2(\mathbf{r}_k) + b_2 \phi_1(\mathbf{r}_k) \tag{29}$$

where

$$a_1 = \pi b \frac{(1 - \alpha_1)(1 + \alpha_2)}{M}$$

$$b_1 = -\pi b\, \eta_2 \frac{(1 - \alpha_2)(3\beta_1 - 2 - \alpha_1\beta_1)}{M}$$

$$a_2 = \pi b \frac{(1 - \alpha_2)[1 + \eta_1(3\beta_1 - 2) + \alpha_1(1 - \eta_1\beta_1)]}{M}$$

$$b_2 = -\pi b\, p \frac{(1 - \alpha_1)(3\beta_2 - 2 - \alpha_2\beta_2)}{M}$$

$$M = [1 + \eta_1(3\beta_1 - 2) + \alpha_1(1 - \eta_1\beta_1)]\,(1 + \alpha_2) - p\eta_2(3\beta_2 - 2 - \alpha_2\beta_2)(3\beta_1 - 2 - \alpha_1\beta_1) \tag{30}$$

Thus, the differential equations (Eqs. 19 and 20) become

$$(-\nabla^2 + \kappa_1^2)\,\phi_1(\mathbf{r}) = \sum_{k=1}^{N} (\phi_{1k})\,\delta(\mathbf{r} - \mathbf{r}_k) \tag{31}$$

$$(-\nabla^2 + \kappa_2^2)\,\phi_2(\mathbf{r}) = \frac{p}{D_2}\Sigma_R\, \phi_1(\mathbf{r}) + \sum_{k=1}^{N}(\phi_{2k})\,\delta(\mathbf{r} - \mathbf{r}_k) \tag{32}$$

where

$$(\phi_{1k}) = \frac{1}{D_1}\{\phi_1(\mathbf{r}_k)\,[\eta_2 b_2 + (\eta_1 - 1)a_1] + \phi_2(\mathbf{r}_k)\,[\eta_2 a_2 + (\eta_1 - 1)\, b_1]\} \tag{33}$$

$$(\phi_{2k}) = \frac{1}{D_2}\{\phi_1(\mathbf{r}_k)\,[pa_1 - b_2] + \phi_2(\mathbf{r}_k)\,[pb_1 - a_2]\} \tag{34}$$

B. Solution of the Balance Equations; Criticality Condition

For an infinite moderator, a solution of Eq. 31 is

$$\phi_1(\mathbf{r}) = \sum_{k=1}^{N} A_k\, K_0(\kappa_1\,|\mathbf{r} - \mathbf{r}_k|) \tag{35}$$

To compute A_k, one again neglects, in the derivative of the flux at a rod k, the derivative of the flux due to the other rods when compared with the derivative of the flux due to the rod k. The current at the boundary of the k element is computed in the same manner as that used in Section II, with the result:

$$A_k = \frac{(\phi_{1k})}{2\pi b\, \kappa_1 K_1(\kappa_1 b)} = (\phi_{1k})\, F \tag{36}$$

where

$$F = \frac{1}{2\pi b\, \kappa_1 K_1(\kappa_1 b)} \tag{37}$$

Then,

$$\phi_1(\mathbf{r}) = F \sum_{k=1}^{N} (\phi_{1k}) K_0(\kappa_1 |\mathbf{r} - \mathbf{r}_k|) \tag{38}$$

By substitution into Eq. 32,

$$(-\nabla^2 + \kappa_2^2)\,\phi_2(\mathbf{r}) = \frac{p}{D_2} \Sigma_R\, F \sum_{k=1}^{N} (\phi_{1k})\, K_0(\kappa_1\, |\mathbf{r} - \mathbf{r}|) + \sum_{k=1}^{N} (\phi_{2k})\, \delta(\mathbf{r} - \mathbf{r}_k) \tag{39}$$

As in Section II, a Fourier-transform procedure is used to solve this Equation, resulting in

$$\begin{aligned}\phi_2(\mathbf{r}) = {} & \frac{p}{D_2} \frac{\Sigma_R F}{\kappa_2^2 - \kappa_1^2} \sum_{k=1}^{N} (\phi_{1k})\, [K_0(\kappa_1\, |\mathbf{r} - \mathbf{r}_k|) - K_0\,(\kappa_2\, |\mathbf{r} - \mathbf{r}_k|)] \\ & + \frac{1}{2\pi} \sum_{k=1}^{N} (\phi_{2k})\, K_0\,(\kappa_2\, |\mathbf{r} - \mathbf{r}_k|)\end{aligned} \tag{40}$$

To obtain the criticality condition, one computes from Eqs. 38 and 39 the thermal and fast fluxes at the surface of each rod:

$$\phi_1(\mathbf{r}_m) = \frac{F}{D_1} \sum_{k=1}^{N} \{\phi_1(\mathbf{r}_k)\, [\eta_2 b_2 + (\eta_1 - 1) a_1] + \phi_2(\mathbf{r}_k)\, [\eta_2 a_2 + (\eta_1 - 1) b_1]\}\, K_0\,(\kappa_1 |\mathbf{r}_m - \mathbf{r}_k|) \tag{41}$$

$$\begin{aligned}\phi_2(\mathbf{r}_m) = {} & \frac{p}{D_2} \frac{\Sigma_R F}{\kappa_2^2 - \kappa_1^2} \frac{1}{D_1} \sum_{k=1}^{N} \{\phi_1(\mathbf{r}_k)\, [\eta_2 b_2 + (\eta_1 - 1) a_1] \\ & + \phi_2(\mathbf{r}_k)\, [\eta_2 a_2 + (\eta_1 - 1) b_1]\}\, \{K_0\,(\kappa_1 |\mathbf{r}_m - \mathbf{r}_k|) - K_0\,(\kappa_2\, |\mathbf{r}_m - \mathbf{r}_k|)\} \\ & + \frac{1}{2\pi} \frac{1}{D_2} \sum_{k=1}^{N} \{\phi_1(\mathbf{r}_k)\, [p a_1 - b_2] + \phi_2(\mathbf{r}_k)\, [p b_1 - a_2]\}\, K_0\,(\kappa_2\, |\mathbf{r}_m - \mathbf{r}_k|)\end{aligned} \tag{42}$$

The result is a system of $2N$ linear homogeneous equations whose $2N$ unknowns are the $\phi_1(\mathbf{r}_k)$ and $\phi_2(\mathbf{r}_k)$. Non-trivial solutions exist only if $\Delta = 0$ (or, the determinant of the system is zero). The criticality condition is that the $2N$-order determinant Δ must vanish.

In practice, use can be made of symmetrically situated fuel elements having the same surface flux $\phi(\mathbf{r}_k)$, and the number of unknowns can thus be considerably reduced. Most often, the order of the determinant is smaller than $2N$.

IV. CALCULATION OF THE FOUR COEFFICIENTS

The four coefficients $\alpha_1\beta_1\alpha_2\beta_2$, introduced in Section III, depend on the nuclear properties and geometrical configuration of the fuel element. Each fuel element, if different from the others, can have a different set of these coefficients. Considered here is a simple model of a fuel element, accompanied by a method of obtaining these coefficients.

The element is composed of two concentric cylindrical regions. The inner region contains moderating material ($\Sigma_a < \Sigma_s$), and the outer region contains fuel ($\Sigma_a \simeq \Sigma_s$). The cross section of the fuel element is shown in Fig. 1.

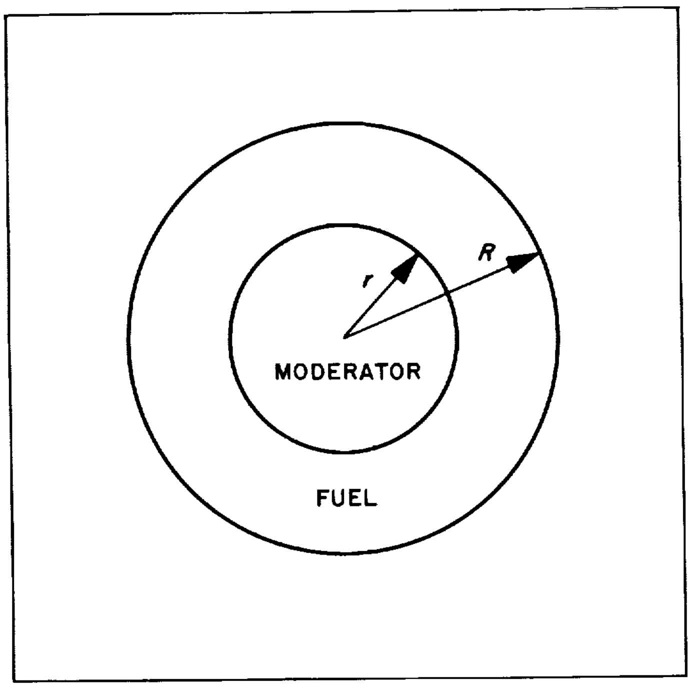

Fig. 1. Cross section of fuel rod containing moderator

First, a few probabilities are defined which will be useful in obtaining these coefficients:

P_1 is the probability that a neutron coming from the inner moderator goes through the fuel shell without making any collision.

P_2 is the probability that a neutron escapes from the fuel element after a scattering collision in the fuel.

P_3 is the probability that a neutron enters the inner moderator after a scattering collision in the fuel.

π_1 is the probability that a neutron coming from outside goes through the fuel shell only and escapes from the fuel element without a collision.

π_2 is the probability that a neutron coming from outside reaches the inner moderator without a collision in the fuel.

π_3 is the probability that a fast neutron (born in the fuel annulus) entering the inner moderator does not thermalize inside the moderator.

π_3' is the probability that a fast neutron (born outside the fuel annulus) entering the moderator does not thermalize inside the moderator.

π_4 is the probability that a thermal neutron escapes the inner moderator.

These probabilities (except π_3, π_3', and π_4) must be defined for both fast and thermal neutrons. In general, accurate calculation of these transmission probabilities is quite difficult and could be the subject of a separate study. For the present purposes, it was considered sufficient to make rough calculations utilizing rather gross simplifying assumptions. These assumptions and the calculations are outlined in Appendix B. The probabilities are defined graphically in Fig. 2.

The following analysis shows how the detailed transmission probabilities are combined to give the four coefficients desired.

A. The Coefficient α_1: Probability That a Fast Neutron Entering the Fuel Element Escapes as a Fast Neutron

Referring to Fig. 2, assume that S fast neutrons enter the fuel element per unit time and per unit length, and that x of these neutrons, after entering the inner moderator, leave it again as fast neutrons. Hence,

$$[S(1-\pi_1-\pi_2)+x(1-P_1)]\,\omega$$

neutrons make at least one scattering collision in the fuel, whereas

$$[S(1-\pi_1-\pi_2)+x(1-P_1)]\,\omega\,(1-P_2-P_3)\,\omega$$

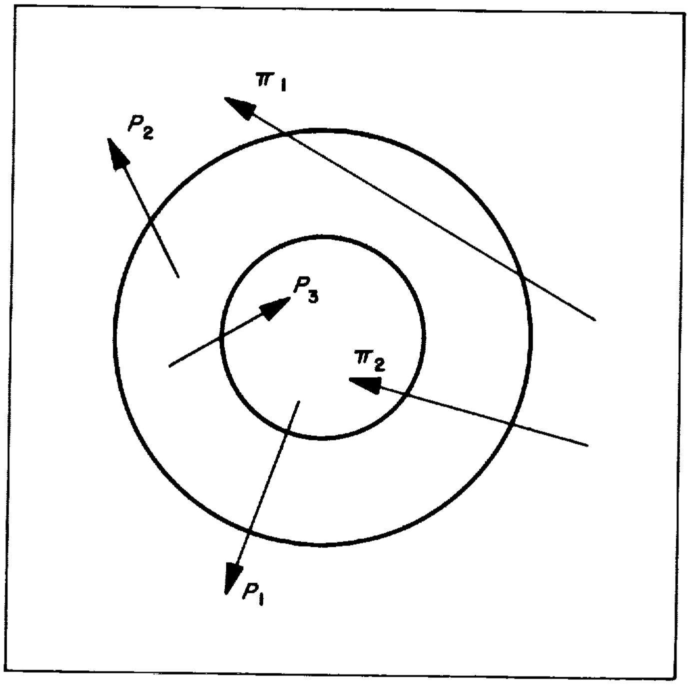

Fig. 2. Escape and transmission probabilities in fuel model

make two scattering collisions in the fuel, and so on. Here, $\omega = \Sigma_s/\Sigma_T$. The total number of scattering collisions these neutrons make in the fuel is (by summation)

$$\frac{[S(1 - \pi_1 - \pi_2) + x(1 - P_1)]\,\omega}{1 - (1 - P_2 - P_3)\,\omega}$$

Thus, the number of escaping neutrons is

$$S\alpha_1 = S\pi_1 + \frac{[S(1 - \pi_1 - \pi_2) + x(1 - P_1)]\,\omega}{1 - (1 - P_2 - P_3)\,\omega} P_2 + xP_1 \tag{43}$$

and the number of neutrons entering the inner moderator is

$$\frac{x}{\pi_3'} = S\pi_2 + \frac{[S(1 - \pi_1 - \pi_2) + x(1 - P_1)]\,\omega}{1 - (1 - P_2 - P_3)\,\omega} P_3 \tag{44}$$

Solving the two previous Equations for α_1 gives

$$\alpha_1 = \pi_1 + \frac{(1 - \pi_1 - \pi_2)\omega P_2}{1 - (1 - P_2 - P_3)\omega} + \epsilon_1 \frac{(P_2 + P_1P_3 - P_1)\omega + P_1}{1 - (1 - P_2 - P_3)\omega} \tag{45}$$

where

$$\epsilon_1 = \pi_3' \frac{\pi_2(1 - P_2 - P_3)\omega\pi_2 + (1 - \pi_1 - \pi_2)\omega P_3}{1 - (1 - P_2 - P_3)\omega - (1 - P_1)\omega P_3 \pi_3'} \tag{46}$$

B. The Coefficient β_1: Probability That a Fast Neutron Born From Fission Inside the Fuel Element Escapes From the Fuel Element as a Fast Neutron

Assume that S neutrons are born from fission in the fuel element per unit length and time, and that x of these, after entering the inner moderator, leave it again as fast neutrons. Then,

$$[S(1 - P_2 - P_3) + x(1 - P_1)]\omega$$

make at least one scattering collision in the fuel. The total number of scattering collisions these neutrons make in the fuel is

$$\frac{[S(1 - P_2 - P_3) + x(1 - P_1)]\omega}{1 - (1 - P_2 - P_3)\omega}$$

Thus,

$$S\beta_1 = SP_2 + \frac{[S(1 - P_2 - P_3) + x(1 - P_1)]\omega}{1 - (1 - P_2 - P_3)\omega} P_2 + xP_1 \tag{47}$$

of these neutrons escape, whereas

$$\frac{x}{\pi_3} = SP_3 + \frac{[S(1 - P_2 - P_3) + x(1 - P_1)]\omega}{1 - (1 - P_2 - P_3)\omega} P_3 \tag{48}$$

neutrons enter the inner moderator. Hence,

$$\beta_1 = P_2 + \frac{(1 - P_2 - P_3)\omega P_2}{1 - (1 - P_2 - P_3)\omega} + \frac{P_3\pi_3}{1 - (1 - P_2 - P_3)\omega - (1 - P_1)\omega P_3\pi_3} \times \frac{P_1 + \omega(P_2 + P_1P_3 - P_1)}{1 - (1 - P_2 - P_3)\omega} \tag{49}$$

C. The Coefficient α_2: Probability That a Thermal Neutron Entering the Fuel Element Escapes From the Fuel Element

One can use here the same procedure as for α_1, where π_4 replaces π_3'. This yields

$$\alpha_2 = \pi_1 + \frac{(1 - \pi_1 - \pi_2)\omega P_2}{1 - (1 - P_2 - P_3)\omega} + \epsilon_2 \frac{(P_2 + P_1P_3 - P_1)\omega + P_1}{1 - (1 - P_2 - P_3)\omega} \tag{50}$$

where

$$\epsilon_2 = \pi_4 \frac{\pi_2 - (1 - P_2 - P_3)\omega\pi_2 + (1 - \pi_1 - \pi_2)\omega P_3}{1 - (1 - P_2 - P_3)\omega - (1 - P_1)\omega P_3\pi_4} \quad (51)$$

D. The Coefficient β_2: Probability That a Thermal Neutron Thermalized in the Fuel Element Escapes From the Fuel Element

Assume that S neutrons are thermalized in the element per unit length and time, and that x of these, after entering the fuel, return into the inner moderator and subsequently escape again. Then, $(S + x)(1 - P_1)\omega$ neutrons make at least one scattering collision in the fuel. The total number of scattering collisions these neutrons make in the fuel is

$$\frac{(S + x)(1 - P_1)\omega}{1 - (1 - P_2 - P_3)\omega}$$

Thus,

$$S\beta_2 = (S + x) P_1 + \frac{(S + x)(1 - P_1)\omega}{1 - (1 - P_2 - P_3)\omega} P_2 \quad (52)$$

neutrons escape from the rod, and

$$\frac{x}{\pi_4} = \frac{(S + x)(1 - P_1)\omega}{1 - (1 - P_2 - P_3)\omega} P_3 \quad (53)$$

neutrons re-enter the inner moderator. The result is

$$\beta_2 = \frac{P_1 + \omega(P_2 + P_1P_3 - P_1)}{1 - (1 - P_2 - P_3)\omega - (1 - P_1)\omega P_3\pi_4} \quad (54)$$

V. CONCLUSIONS

This development of the heterogeneous method, using the four-coefficient and two-group approach, extends the area of application of the method to include a greater variety of elements, with no expected decrease in the order of accuracy. The same characteristics are present as in the single-coefficient technique; the increased complexity is achieved at the cost of increased computation time.

The size of the criticality determinant is twice that of the same determinant in the single-coefficient case. It is therefore advisable, before using this technique on a complex problem, to examine the array for all possible symmetries. In a regular hexagonal lattice, for example, reduction of the magnitude of the determinant by a factor of 12 is possible.

As presented, the calculation of the individual transmission probabilities is a separate problem and can be performed to any order of precision and complexity required. Multigroup-cell theory or Monte Carlo methods could be utilized if necessary, rather than the simple model presented here.

The flux distribution in the moderator can be obtained from Eqs. 38 and 40. These Equations include the terms (ϕ_{1k}) and (ϕ_{2k}), defined in Eqs. 33 and 34. They depend on $\phi_1(\mathbf{r}_k)$ and $\phi_2(\mathbf{r}_k)$, the fluxes at the surface of each fuel element. The latter are solutions of the homogeneous linear equations (Eqs. 41 and 42), whose determinant has been made zero. Hence, the solution is not unique. A value for one $\phi(\mathbf{r}_k)$ must be arbitrarily chosen. By substituting this value in Eqs. 41 and 42, one gets a system of linear non-homogeneous equations. One of these equations is redundant, because one more equation than unknowns remains. This non-homogeneous system can be solved by calculating determinants of order $2N - 1$, at most.

A program has been drawn up for the IBM 7090 computer which will accomplish both calculations of the coefficients and solution of the determinants.

Further improvements can be made in the four-coefficient method. As was pointed out above, the resonances in the fuel are taken into account by an overall resonance-escape probability. Actually, since the resonance absorptions take place mainly in the fuel, it might be worthwhile to take into account the localization of this phenomenon. This might be done, for instance, by introducing a third group. One observes that the resonance absorptions occur primarily in a relatively narrow range of energy for uranium (between about 2 and 200 ev). Using the flux distribution of this intermediate group, it is possible to determine at what rate these neutrons flow into each fuel element; also, it is possible to evolve a set of cross sections for this group which includes resonance properties.

The introduction of three groups requires the use of nine coefficients, rather than the four employed in the method outlined in Section III. These nine coefficients relate the three sink terms to the three fluxes at the surface of each fuel element and are equivalent to the four coefficients defined in Eqs. 28 and 29.

The next improvement would be to consider a finite medium, which is necessary for the case of thin reflectors. For example, one might apply the refinements developed in the present investigation to the Jonsson theory (Ref. 4).

It should not be forgotten that all these extra effects, if taken into account, give rise to mathematical complications. For instance, the size of the determinant of the criticality condition is tripled if three groups are used. Hence, depending on the number of fuel elements to be used, a compromise must be found between the degree of complexity one can afford and the accuracy one desires.

APPENDIX A

The Thermal Constant in Cylindrical Geometry

The thermal constant γ of a slug in a diffusing medium is defined as the ratio of the net flow per unit time of thermal neutrons into the slug to the value of the thermal flux at the surface of the slug.

One way of obtaining an approximate value of this constant is presented here, based on the following assumptions:

(1) The fuel rod consists of only one kind of material.

(2) The angular distribution of neutrons entering the slug is isotropic (a first-order correction to this approximation is performed here).

(3) The collision density is constant inside the fuel.

The thermal coefficient is related to the transmission coefficient τ, or the fraction of all neutrons incident upon the surface of a lump which pass through the lump without being absorbed:

$$\tau = \frac{j^+}{j^-}$$

$$\gamma = \frac{-2\pi b J}{\phi} = \pi b \frac{j^- - j^+}{j^- + j^+} = \pi b \frac{1-\tau}{1+\tau} \qquad \text{(A-1)}$$

The terms J and ϕ are the net current and the flux, respectively, at the surface of the rod; j^+ and j^- are the partial currents at the same surface (the j^+ current going outward); b is the radius of the slug, which is assumed to be circular cylindrical. From this relationship, τ is computed, and λ is deduced.

1. The Transmission Coefficient τ

Among S neutrons entering the slug, some scatter, some are absorbed, and some escape without undergoing collisions. In paragraph 2 of this Appendix, a computation is made to determine λS, the number of neutrons which escape without undergoing any collision. The number of neutrons $(1-\lambda)$ S, therefore, make at least one collision each. Let

Σ_T = the total macroscopic cross section of the material inside the lump

Σ_s = the macroscopic scattering cross section of the same material

Thus, among the $(1-\lambda)$ S neutrons which make collisions, $(1-\lambda)(\Sigma_s/\Sigma_T)S$ make scattering collisions. In paragraph 3 of this Appendix, a computation method is given for ζ, the probability that those neutrons which have made scattering collisions then escape from the slug. For that calculation, the two following assumptions are made:

(1) In the laboratory system, the scattering is isotropic in the slug material. This assumption is acceptable for heavy nuclei where $\bar{\mu}_0 = 2/3A$. The average cosine of the scattering angle becomes small when compared to unity.

(2) The collision density in the slug is independent of position.

Once ζ is found, it is seen that

$$(1-\lambda)\frac{\Sigma_s}{\Sigma_T}(1-\zeta)\,S$$

of the incoming neutrons make at least two collisions each, and that

$$(1-\lambda)\frac{\Sigma_s}{\Sigma_T}(1-\zeta)\frac{\Sigma_s}{\Sigma_T}S$$

neutrons make at least two scattering collisions each. By the same procedure, it is evident that

$$(1-\lambda)\frac{\Sigma_s}{\Sigma_T}\left[(1-\zeta)\frac{\Sigma_s}{\Sigma_T}\right]^n S$$

neutrons make at least $(n+1)$ scattering collisions each, and that, at each scattering generation, a fraction ζ of the scattering neutrons escapes from the slug.

Thus, the total number of neutrons which escape from the slug after any number of collisions is expressed by

$$S_T = \left[\lambda + (1-\lambda)\frac{\Sigma_s}{\Sigma_T}\zeta + (1-\lambda)\frac{\Sigma_s}{\Sigma_T}(1-\zeta)\frac{\Sigma_s}{\Sigma_T}\zeta + (1-\lambda)\frac{\Sigma_s}{\Sigma_T}(1-\zeta)^2\left(\frac{\Sigma_s}{\Sigma_T}\right)^2\zeta + \cdots\right]S$$

$$= \left[\lambda + \frac{(1-2)\,\zeta\,\Sigma_s}{\Sigma_T - (1-\zeta)\,\Sigma_s}\right]S \qquad \text{(A-2)}$$

Hence, from Eq. A-1, the thermal constant is

$$\gamma = \pi b \frac{(1-\lambda)(\Sigma_T - \Sigma_s)}{(1+\lambda)(\Sigma_T - \Sigma_s) + 2\zeta\Sigma_s} \qquad \text{(A-3)}$$

2. The Probability λ

The term λ is the probability that a neutron entering the slug will escape without making any collision. This probability is equivalent to the transmission coefficient τ of a lump, computed in a first-flight approximation (Ref. 5, p. 247); however, the total cross section is used rather than the absorption cross section.

a. Expression for the probability λ. Needed here is the angular distribution of the velocities of the thermal neutrons impinging upon the surface of the slug (Fig. A-1). A first-order correction to the isotropic distribution is given by the diffusion theory (Ref. 5, p. 171):

$$j^-(\mu,\phi)\,d\mu d\phi = \frac{\mu d\mu d\phi}{4\pi}\left[\phi(0) + \frac{|\nabla\phi(0)|\cos\beta}{\Sigma_s^{(m)}}\right] \qquad \text{(A-4)}$$

where

μ is the cosine of the angle which the direction of motion of the incident neutron makes with the normal to the surface of the lump.

ϕ is the azimuth of the neutron direction about the normal.

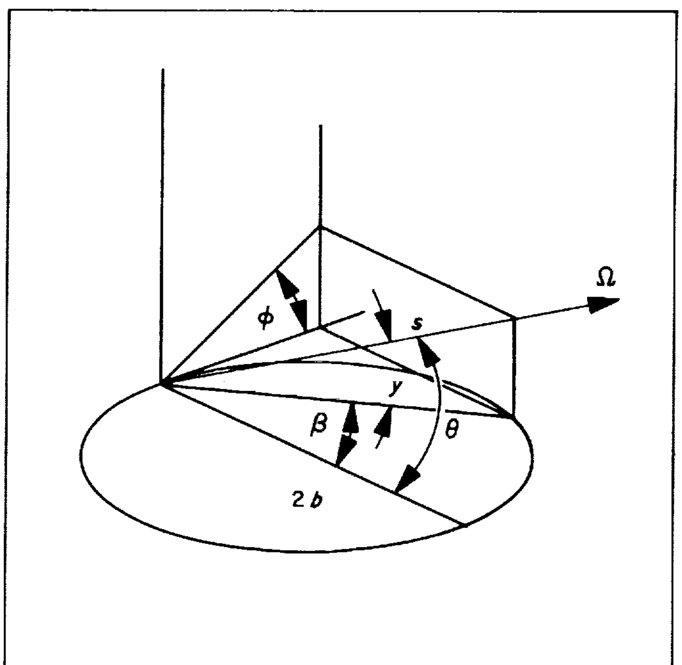

Fig. A-1. Projected path of neutron impinging on surface of fuel slug

$\Sigma_s^{(m)}$ is the scattering cross section of the diffusion medium outside the lump.

β is the angle between the direction of motion of the incident neutron and the flux gradient.

In the present case, the flux gradient can be considered as always oriented perpendicular to the surface of the slug, neglecting the gross variation of the flux in the reactor when compared with the local variation at the slug boundary.

Hence, $\cos\beta \simeq \mu$, and the neutron angular distribution is

$$j^-(\mu,\phi)\,d\mu d\phi = \frac{\mu d\mu d\phi}{4\pi}\left[\phi(0) + \frac{|\nabla\phi(0)|\mu}{\Sigma_s^{(m)}}\right] \qquad \text{(A-5)}$$

where $\nabla\phi(0)$ can be approximated from a diffusion-theory calculation.

For cylindrical geometry,

$$\frac{\nabla\phi(0)}{\phi(0)} = \frac{\phi'(0)}{\phi(0)} \simeq \frac{\kappa I_1(\kappa b)}{I_0(\kappa b)} \qquad \text{(A-6)}$$

where the slug-material constants are expressed by

$$\kappa^2 = \frac{\Sigma_a^{(F)}}{D^{(F)}} \qquad \text{(A-7)}$$

Hence,

$$j^-(\mu,\phi)\,d\mu d\phi = \frac{\mu d\mu d\phi}{4\pi}\phi_0(1 + g\mu) \qquad \text{(A-8)}$$

where

$$g = \frac{\kappa}{\Sigma_s^{(m)}}\frac{I_1(\kappa b)}{I_0(\kappa b)} \qquad \text{(A-9)}$$

The integral, over all directions into the lump, of the velocity distribution is

$$\frac{\phi_0}{4\pi}\int_0^1 \mu(1+g\mu)\,d\mu \int_0^{2\pi} d\phi = \frac{\phi_0}{4}\left(1 + \frac{2g}{3}\right) \qquad \text{(A-10)}$$

The probability that a neutron will pass through a distance of material without making any collision is $e^{-\Sigma_T s}$, where Σ_T is the total cross section of the material, and s is the path length shown in Fig. A-1.

The probability λ can now be expressed as

$$\lambda = \frac{\iint j(\mu,\phi)\, e^{-\Sigma_T s(\mu,\phi)}\, d\mu d\phi}{\iint j(\mu,\phi)\, d\mu d\phi} \qquad \text{(A-11)}$$

$$\lambda = \frac{1}{\pi\left(1+\frac{2g}{3}\right)} \int_0^1 \mu(1+g\mu)\,d\mu \int_0^{2\pi} e^{-\Sigma_T s(\mu,\phi)}\,d\phi \tag{A-12}$$

b. Calculation of the probability λ in cylindrical geometry from Eq. A-12. Referring to Fig. A-1, one makes the change of variables defined by

$$\begin{aligned} \mu &= \cos\beta\cos\gamma \\ \tan\phi &= \frac{\tan\gamma}{\sin\beta} \\ s &= 2b\,\frac{\cos\beta}{\cos\gamma} \end{aligned} \tag{A-13}$$

The Jacobian of the transformation is

$$J = \cos\gamma$$

Hence,

$$\begin{aligned}\lambda = &\frac{4}{\pi\left(1+\frac{2}{3}g\right)} \\ &\times \left\{ \iint_0^{\pi/2} \cos\beta\cos^2\gamma\, e^{-\Sigma_T 2b(\cos\beta/\cos\gamma)}\,d\beta\,d\gamma \right. \\ &\left. + g\iint_0^{\pi/2} \cos^2\beta\cos^3\gamma\, e^{-\Sigma_T 2b(\cos\beta/\cos\gamma)}\,d\beta\,d\gamma \right\}\end{aligned} \tag{A-14}$$

These integrals were evaluated numerically. Writing,

$$\lambda = \frac{4}{\pi\left(1+\frac{2}{3}g\right)}(I_1 + gI_2)$$

the values of I_1 and I_2 are plotted in Fig. A-2 as functions of $\Sigma_T b$.

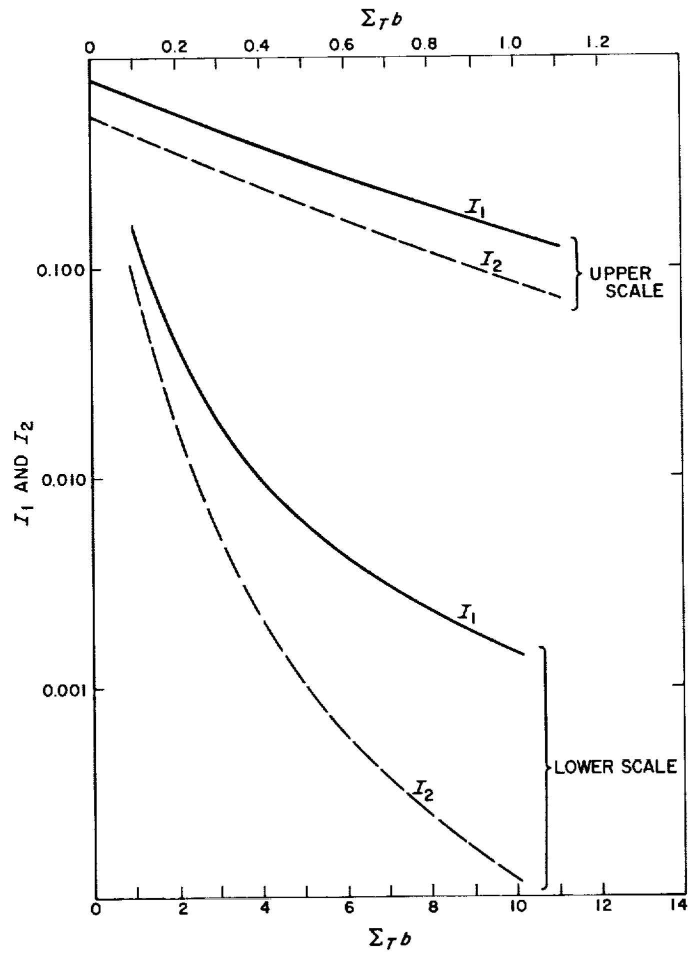

Fig. A-2. Numerical values of I_1 and I_2 vs fuel-rod radius measured in mean free paths

3. The Probability ζ

The term ζ is the probability that a neutron will escape from the cylindrical slug after a scattering collision. It is assumed that the scattering is homogeneous and isotropic inside the slug. Hence, the system is equivalent to a uniform-source material whose shape is a circular cylinder, and ζ is the probability that a neutron born in this source will escape.

This problem has been treated by Cohen and Estabrook (Ref. 6), with the following result:

$$\begin{aligned}\zeta = \frac{2\Sigma_T b}{3}\Big\{ &-2 + \left(2\Sigma_T b + \frac{1}{\Sigma_T b}\right) I_1(\Sigma_T b)\,K_1(\Sigma_T b) \\ &+ I_0(\Sigma_T b)\,K_1(\Sigma_T b) - I_1(\Sigma_T b)\,K_0(\Sigma_T b) \\ &+ 2\Sigma_T b\, I_0(\Sigma_T b) K_0(\Sigma_T b)\Big\}\end{aligned} \tag{A-15}$$

APPENDIX B

Approximate Method for Evaluating Neutron-Transmission Probabilities

In computing the transmission probabilities, the following approximations are made:

(1) The scattering is isotropic in the laboratory system.

(2) The angular distribution of neutrons impinging on the outside surface of the fuel element is isotropic.

(3) The collision density $\Sigma_T \phi$ in each region of the fuel element is space-independent for both thermal and fast neutrons.

1. The Probability P_1

The term P_1 designates the probability that a neutron coming from the inner moderator goes through the fuel shell without making any collision. It is assumed, as a first approximation, that the angular distribution of the neutrons entering the moderator is uniform. The probability that a neutron coming out of the element of area dA (Fig. B-1) through a solid angle $d\Omega$ about θ and ϕ goes through the fuel shell without colliding is $e^{\Sigma_T s_0}$ where Σ_T is the total cross section of the fuel. With use of the approximation above, it follows that the expression

$$j_1^+ \, d\Omega = \frac{1}{\pi} \cos^2\theta \cos\phi \, d\phi \, d\theta$$

is the probability that a neutron going through dA is in a solid angle $d\Omega$ about θ and ϕ ($1/\pi$ is a normalizing factor). The product $e^{-\Sigma_T s_0} j_1^+ \, d\Omega$ is the probability that a neutron going through dA is in a solid angle $d\Omega$ and penetrates the fuel shell without a collision. The probability P_1, the sum of these elementary probabilities over the solid angle, is given by

$$P_1 = \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2\theta \cos\phi \, e^{-\Sigma_T s_0} \, d\phi d\theta \qquad \text{(B-1)}$$

where, according to Fig. B-1,

$$s_0 = \frac{R}{\cos\theta}\left(\sqrt{1 - \frac{r^2}{R^2}\sin^2\phi} - \frac{r}{R}\cos\phi\right)$$

One may then write

$$P_1 = \frac{4}{\pi} Z\left(\frac{r}{R}, R, \Sigma_T\right) \qquad \text{(B-2)}$$

where

$$Z\left(\frac{r}{R}, R, \Sigma_T\right) = \int_0^{\pi/2} \int_0^{\pi/2} \cos^2\theta \cos\phi \, e^{-\Sigma_T (R/\cos\theta)\left[\sqrt{1-\left(\frac{r}{R}\right)^2 \sin^2\phi} - \frac{r}{R}\cos\phi\right]} \, d\phi \, d\theta \quad \text{(B-3)}$$

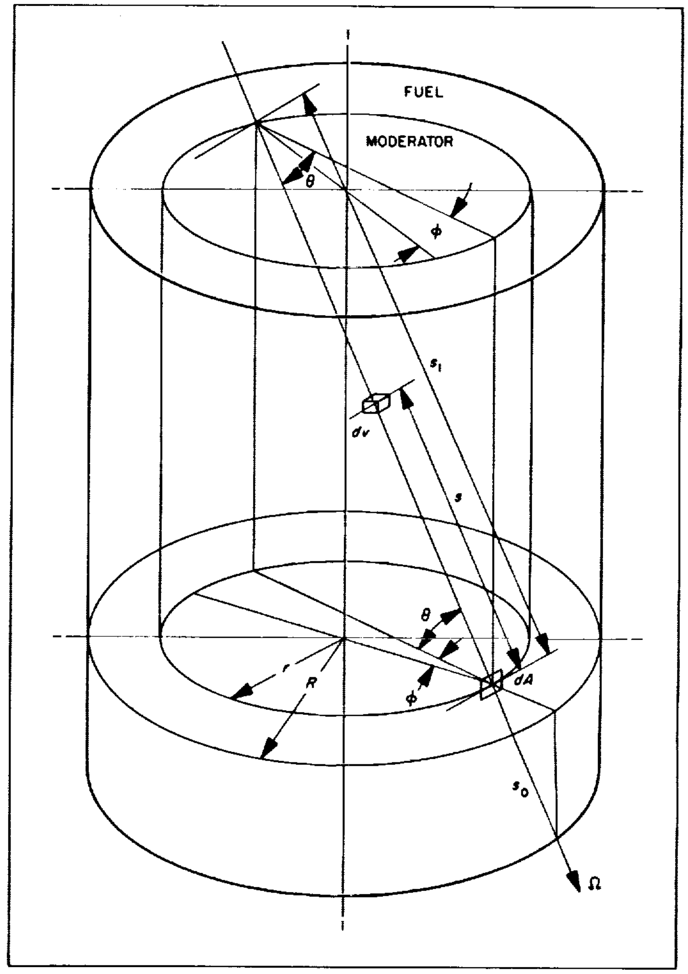

Fig. B-1. Three-dimensional path of neutron from inner moderator passing through fuel shell without collision

2. The Probability P_2

The probability that a neutron escapes from the fuel element after a scattering collision in the fuel is denoted by the term P_2. On the assumption that scattering is isotropic in the fuel, and that the flux can be considered as a constant inside the fuel, each volume element dv is considered as a unit source. The probability that a neutron coming from dv reaches an area dA on the outside surface of the fuel element is the product of $e^{\Sigma_T s}$ and the fraction of solid angle through which dA is seen from dv, or

$$\frac{dA \cos\theta \cos\phi}{4\pi s^2}$$

where s is the distance between dA and dv.

To obtain P_2, this product is integrated over the volume and divided by the source strength, which is v:

$$P_2 = \frac{1}{v}\int_A \int_s \int_\theta \int_\phi \frac{dA \cos\theta \cos\phi}{4\pi s^2} e^{-\Sigma_T s} s^2 \, ds \cos\theta \, d\phi \, d\theta \tag{B-4}$$

Note that there are two kinds of limits of integration, depending on whether ϕ is larger or smaller than α (Fig. B-2) where $\alpha = \sin^{-1}(r/R)$. These two limits are

$$s_2 = 2R \frac{\cos\phi}{\cos\theta}$$

$$s_3 = \frac{R}{\cos\theta}\left(\cos\phi - \sqrt{\frac{r^2}{R^2} - \sin^2\phi}\right)$$

Therefore, the resulting expression is

$$P_2 = \frac{2R}{\Sigma_T \pi (R^2 - r^2)}\left\{\frac{\pi}{4} - I\left(\frac{r}{R}, R, \Sigma_T\right) - W\left(\frac{r}{R}, R, \Sigma_T\right)\right\} \tag{B-5}$$

where

$$I\left(\frac{r}{R}, R, \Sigma_T\right) = \int_0^{\pi/2}\int_{\phi=\alpha}^{\pi/2} \cos^2\theta \cos\phi \, e^{-\Sigma_T 2R(\cos\phi/\cos\theta)} \, d\phi \, d\theta \tag{B-6}$$

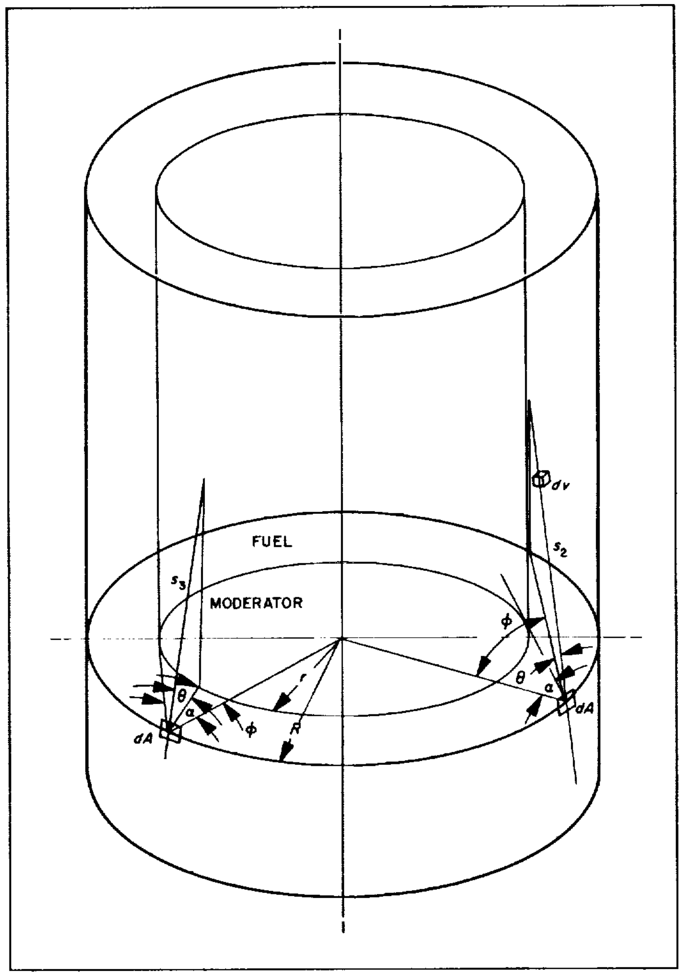

Fig. B-2. Three-dimensional paths of neutrons escaping from fuel through outer wall

3. The Probability P_3

The term P_3 represents the probability that a neutron enters the inner moderator after a scattering collision in the fuel. Considered now is a small volume element dv in the fuel and a small area dA on the surface of the inner moderator. Applying the same reasoning as for P_2, the limit of integration (Fig. B-3) is now

$$s_4 = \frac{R}{\cos\theta}\left(\sqrt{1 - \frac{r^2}{R^2}\sin^2\phi} - \frac{r}{R}\cos\phi\right)$$

$$W\left(\frac{r}{R}, R, \Sigma_T\right) = \int_0^{\pi/2}\int_{\phi=0}^{\alpha} \cos^2\theta \cos\phi \, e^{-\Sigma_T (R/\cos\theta)\left[\cos\phi - \sqrt{\left(\frac{r}{R}\right)^2 - \sin^2\phi}\right]} \, d\phi \, d\theta \tag{B-7}$$

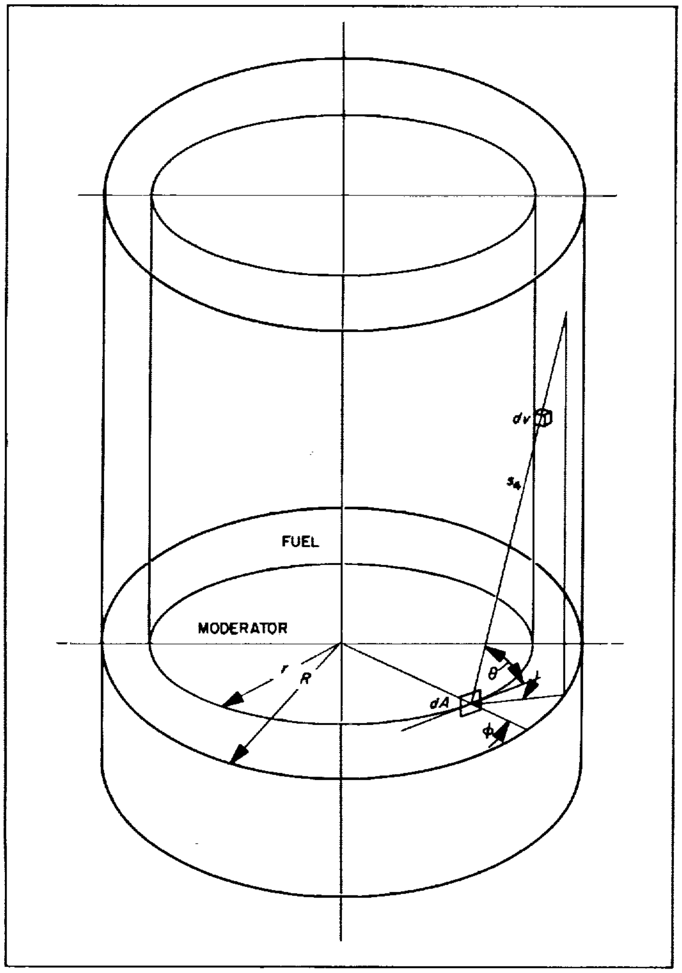

Fig. B-3. Three-dimensional path of neutron escaping from fuel to inner moderator

and the result of the integration is

$$P_3 = \frac{2r}{\Sigma_T \pi (R^2 - r^2)} \left\{ \frac{\pi}{4} - Z\left(\frac{r}{R}, R, \Sigma_T\right) \right\} \qquad \text{(B-8)}$$

where $z\ (r/R,\ R,\ \Sigma_T)$ is defined by Eq. B-3.

4. The Probability π_1

The probability that a neutron coming from outside goes through the fuel shell only and escapes from the fuel element without a collision is represented by π_1. Assume an isotropic angular distribution for the velocities of the neutrons coming into the fuel element,

$$j^-(\theta, \phi)\, d\phi\, d\theta = \cos^2\theta \cos\phi\, d\phi\, d\theta \qquad \text{(B-9)}$$

The probability that a neutron traveling in a solid angle $d\Omega$ about θ and ϕ escapes is $e^{-\Sigma_T s_2}$, provided that $\phi > \alpha$ (see Fig. B-2). Hence,

$$\pi_1 = \frac{\iint j^-(\theta, \phi)\, e^{-\Sigma_T s_2} d\phi\, d\theta}{\iint j^-(\theta, \phi)\, d\phi\, d\theta}$$

Since

$$s_2 = 2R \frac{\cos\phi}{\cos\theta}$$

then,

$$\pi_1 = \frac{4}{\pi} I \left(\frac{r}{R}, R, \Sigma_T\right) \qquad \text{(B-10)}$$

where $I\ (r/R,\ R,\ \Sigma_T)$ is defined by Eq. B-6.

5. The Probability π_2

The term π_2 denotes the probability that a neutron coming from outside reaches the inner moderator without a collision in the fuel. Using the same procedure as for π_1, one obtains

$$\pi_2 = \frac{\iint j^-(\theta, \phi)\, e^{-\Sigma_T s_3} d\phi\, d\theta}{\iint j^-(\theta,\phi)\, d\phi\, d\theta}$$

where, according to Fig. B-2,

$$s_3 = \frac{R}{\cos\theta}\left(\cos\phi - \sqrt{\frac{r^2}{R^2} - \sin^2\phi}\right)$$

Therefore,

$$\pi_2 = \frac{4}{\pi} W \left(\frac{r}{R}, R, \Sigma_T\right) \qquad \text{(B-11)}$$

where $W\ (r/R,\ R,\ \Sigma_T)$ is defined by Eq. B-7.

6. The Probability π_3

The term π_3 expresses the probability that a fast neutron entering the inner moderator does not thermalize inside the moderator. Two cases are distinguished:

(1) The neutrons born inside the slug from fission. These neutrons have a well known average lethargy, and one can have an idea of the average number of scattering collisions which will make these neutrons thermal.

(2) The fast neutrons which enter the slug from outside. These neutrons belong to the fast group, but actually their lethargy is not well defined and is spread between thermal and minimum lethargies.

In case (1), knowing the probability that a neutron escapes at each scattering collision, one can determine the probability that a fast neutron escapes without becoming thermal. From the notations and results of Appendix A,

λ = the probability that a neutron going through the moderator does not collide.

ζ = the probability that a neutron, after a scattering collision in the moderator, escapes from the moderator.

In this case, it can be assumed that $\Sigma_s/\Sigma_T = 1$, because only fast neutrons are considered.

Calling N_{avg} the average number of collisions which make a fission neutron thermal, one obtains

$$\pi_3 = \lambda + (1-\lambda)\,\zeta + (1-\lambda)\,(1-\zeta)\,\zeta + \cdots$$
$$\cdots + (1-\lambda)\,(1-\zeta)^{N-2}_{avg}$$
$$\pi_3 = \lambda + (1-\lambda)\,\zeta\,\frac{1-(1-\zeta)^{N-1}_{avg}}{1-(1-\zeta)}$$
$$= \lambda + (1-\lambda)\,\left[1-(1-\zeta)^{N-1}_{avg}\right] \qquad \text{(B-12)}$$

Turning to case (2), in the context of the two-group model, one can say that at each scattering collision the average probability that a fast neutron becomes thermal is Σ_R/Σ_s, neglecting fast absorptions in the moderator. Therefore, using the results of Appendix A,

$$\pi_3' = \lambda + (1-\lambda)\left(1-\frac{\Sigma_R}{\Sigma_s}\right)\frac{\zeta}{1-(1-\zeta)\left(1-\dfrac{\Sigma_R}{\Sigma_s}\right)} \qquad \text{(B-13)}$$

In these expressions,

$$\lambda = \frac{4}{\pi}\,I\,(0, r, \Sigma_T)$$

$$\zeta = \frac{2l}{3}\left\{-2+\left(2l+\frac{1}{l}\right)I_1(l)\,K_1(l) + I_0(l)\,K_1(l) - I_1(l)\,K_0(l) + 2l\,I_0(l)\,K_0(l)\right\}$$

where

$$l = \Sigma_T\, r$$

7. The Probability π_4

The probability that a thermal neutron escapes the inner moderator is represented by π_4. This probability accounts for the thermal absorption which takes place in the moderator. Most of the time, it is very close to 1. The term π_4 is equivalent to the transmission coefficient of the moderator slug. Hence, according to the results of Appendix A,

$$\pi_4 = \lambda + (1-\lambda)\,\frac{\Sigma_s}{\Sigma_T}\,\frac{\zeta}{1-(1-\zeta)\,\dfrac{\Sigma_s}{\Sigma_T}} \qquad \text{(B-14)}$$

where Σ_s/Σ_T of the moderator is used.

NOMENCLATURE

b radius of a fuel element

B_z longitudinal buckling

D_1, D_2 fast and thermal diffusion constants

h half-length of a fuel element

$K_n(x)$ modified Bessel function of the second kind and of the nth order

N total number of fuel elements

N_1 average number of collisions to make a fission neutron thermal

p resonance-escape probability

r radius of the inner moderator cylinder of a fission-electric cell element

$\mathbf{r}$ two-dimensional vector: space variable in transverse flux equations after removal of the z dependency

R outside radius of the fuel layer of a fission-electric cell element

$S_k^{(1)}$, $S_k^{(2)}$ fast and thermal sink strengths of the singularity

NOMENCLATURE (Cont'd)

α $\sin^{-1}(r/R)$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ four coefficients connecting sink strengths to fluxes

γ thermal constant

Δ_z reflector savings on one end along the z axis

ϵ scale factor

ζ probability that a neutron, after colliding in the fuel shell, escapes without making an extra collision

η_1 average number of neutrons produced per fast absorption in the fuel element

η_2 average number of neutrons produced per thermal absorption in the fuel element

κ transverse component of the inverse of diffusion length

λ probability that a neutron going through the fuel shell does not collide

$\Sigma_a^{(1)}, \Sigma_a^{(2)}$ fast and thermal macroscopic absorption cross sections of the pertinent material

Σ_R macroscopic removal cross section of the moderator

Σ_s macroscopic scattering cross section of the pertinent material

Σ_T macroscopic total cross section of the pertinent material

$\phi_1(\mathbf{r})$ fast flux at point $\mathbf{r}$

$\phi_2(\mathbf{r})$ thermal flux at point $\mathbf{r}$

$\phi_1(\mathbf{r}_k), \phi_2(\mathbf{r}_k)$ flux at the surface of the fuel element

$(\boldsymbol{\rho})$ three-dimensional vector: space variable in overall flux equations

τ transmission coefficient of a slug

ω ratio of the scattering and total cross sections in the pertinent material

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